INFINIT SERIES IN P ADIC

MR. ANKIT KUMAR YADAV , AKTU LUCKNOW

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1 Introduction

In this paper we be talk about infinit series how to deal it .

1.1 history

It wasn't obvious to the Greeks that there are numbers that couldn't be written as a ratio of two integers. Pythagoras and his followers were deep admirers of numerical ratios and studied their existence in the world, as they arose in music, geometry, cosmology, and the body. In fact, they believed that all numbers could be represented as a ratio — what we call a rational number — and the existence of an irrational number was seen as an affront to the gods.Legend says that in 470 B.C. when Hippasus proved the irrationality of $\sqrt{2}$ he was drowned for spreading such absurd truths!

The most natural way we, as humans, define the distance between two numbers is the Euclidean metric d(x, y) = $|x - y|$ as it corresponds to the distance between x and y on the real number line. As we have seen, we can use this metric to complete Q into the field of real numbers R . But what if we define a different way to describe the distance between two rational numbers?

we constructed R from Q by completing the field Q with respect to the Euclidean metric, we can now define a p-adic number to be an equivalence class of Cauchy sequences under the p-adic metric

2 Definition

Here we describe the p-adic metric ,

he set of all equivalence classes of Cauchy sequences under the p-adic metric is the field Q_p of p-adic numbers. The field Q_p is the completion of the normed field Q using the p-adic metric.

Let K be a field that is complete with respect to a nontrivial nonarchimedean absolute value ||, such as Q_p with its absolute value $||_p$. We will look at power series over K(that is, with coefficients in K) and a variable x coming from K.

2.1 Definition

the p-adic norm can be defined

Let p be a prime number. Define the function $f : Z \to R$ by

$$|x|_p = \{\ 1p^{\mathrm{val}_p(x)}, x \neq 00, x = 0$$

where $\mathrm{val}_p(x)$ denotes the p-adic valuation of x, i.e., the highest exponent $k \in N$ such that $p^k \mid x.$, x = 0 $mod(p^k)$

major course in Real analysis is the representation of functions as power series

$\sum_{n\geq 0} a_n x^n$,

where the coefficients a_n are real number and x are vareable $x \in R$, In this paper we will talk about power series and devloped power series .

(1) there is convergence at $x \in K$ when $|x| < R$ and not when $|x| > R$

(2) A power series is uniformly continuous on each closed disc (of positive radius) where it converges.

ower series can be differentiated termwise in its disc of convergence

(3) Power series can be differentiated term-wise in its of convergence .

There are some Contrasts ;

1) The convergence of a power series at its radius of convergence R does not exhibit mixed behavior: it is either convergent at all x with —x— = R or not convergent at all x with $|x|$ = R, unlike $\sum_{n\geq 1} x^n/n$ on R, where R = 1 and there is convergence at x = 1 but not at x = 1.

(2) A power series can be expanded around a new point in its disc of convergence and the new series has exactly the same disc of convergence as the original series. OverR, a recentered power series often converges at some numbers outside the interval of convergence of the original series.

2.2 Generalities on infinite series

Before discussing power series, we set out a few general properties of infinite series overK. Since our eventual goal is power series, we will write infinite series as $\sum_{n\geq 0} \mathrm{a}_n$, with the indexing starting at n = 0 instead of n = 1.

The most basic property of an infinite series$\sum_{n\geq 0} a_n$ over K is that it converges if and only if $|a_n| \to 0$: convergence implies the general term tends to 0 by the same reasoning as with infinite series over R, and conversely the general term tending to 0 makes the sequence of partial sums a Cauchy sequence in K by the strong triangle inequality, so the series converges in K by completeness .

3 Theorem's

3.1 Theorem

If $a_n \to 0$ in K $|\sum_{n\geq 0} a_n| \geq max_{n\geq 0}\ |a_n|$.

proof. Let $s_n = a_0 + a_1 + a_2 + a_3 + ... + a_{n-1} + a_n$be the nth partial sum and $s_n = \sum_{n\geq 0} a_n = \lim_{n\to\infty} a_n if s_n$ get 0 then we have , s = 0

$|\sum_{n\geq 0} a_n| = 0 \geq \max|a_n|$

if $s_n \neq 0$ then we have $|s| = |s_N|$, $N >> 0$ then that is why

$$s_n$$

has only finit terms $|s_N| = |\sum_{n=0}^{N} a_n| = |a_0 + ... + a_N| = \max(|a_0|, , , , , , |a_N|) \geq max_{n\geq 0}|a_n|$.

3.2 Theorem

we have $\sum_{n\geq 0} a_n$ and $\sum_{n\geq 0} b_n$ converge then $\sum_{n\geq 0}(a_n + b_n) = \sum_{n\geq 0} a_n + \sum_{n\geq 0} b_n$ for each $c \in K$, $\sum_{n\geq 0} ca_n = c\sum_{n\geq 0} a_n$

proof. Let $A = \sum_{n\geq o} a_n$ and $B = \sum_{n\geq 0} b_n$ here the $\sum_{n=0}^{N} a_n \to A$ and $\sum_{n=0}^{N} b_n \to B$ as N$\to \infty$ as know I the additive and multiplicative Propertiy

$\sum_{n=0}^{N}(a_n + b_n) \to$ A+B

or

$\sum_{n=0}^{N} ca_n \to$cA .

3.3 theorem

. If an is a sequence in K and there is a sequence $b_n$in R such that $b_n \geq |a_n|$ for $n >> 0$, then convergence of $\sum_{n\geq 0} b_n$ in R implies convergence of $\sum_{n\geq 0} a_n$ in K.

3.3.1 Example

This example is due to the infinity series

$S = \sum_{n\geq 1}(-1)^{n-1} = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3}.....nowherewehave, S' = 1\text{-}\frac{1}{2}\text{-}\frac{1}{4}+\frac{1}{3}.....$ add each positive and negative the we get,

$S' = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10}.....$

$S' = \frac{1}{2}[1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{4}.....]$

S' = S/2

half of harmonic series rearrange the terms we often new infinity series .

4 Divisibility by 2 or 3

Moduli 2 or 3 [introduce by ramanujan theta function $\varphi(q)$ and ϕ(q) , Denote by

$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}$,

$\phi = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$

[jacobi's triple product identity]

consider Higher moduli for we shall analogous formulae are in the cases of moduli 2 and 3 . it is easy to see that

$\frac{(q^4;q^4)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$ +2J = $\phi(q) + 2J$

it follows that

p(n)-p(n-4)-p(n-8)+p(n-20) + p(n-28)-.....

its odd or even according as n is a triangular number or not where 4,8,20,28.... are number of the form 2v(3v+1) and 2v(3v-1).

4.1 Ramanujan Divisibility of $\tau(n)$

now here q¿0,

$(q;q)_\infty^{24} = \frac{(q^2;q^2)_\infty^8}{(q;q^2)_\infty^8}$

q$\phi^8(q) = \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-qn^2}$

and $\sum n = 1^\infty n^4 q^n = \frac{q}{1-q^2} + 16J$

$\sum_{n=1}^{\infty} n^8 q^n = \frac{q}{1-q^2} + 32J$

[since

$\sum_{n=1}^{\infty} n^4 q^n \equiv 1.q + 0.q^2 + 1.q^3 + 0.q^4 + ... = \frac{q}{1-q^2}(mod16)$

congerence theorem $a \equiv b(modn)$ as written in the form a-b=nk here k is as constant.

$\sum_{n=1}^{\infty} n^8 q^n = 1.q + 0.q^2 + 1.q^3 + 0.q^4 ..... = \frac{q}{1-q^2}(mod32)$

since ,

n$^4 \equiv 0, 1(mod16)$

now here i can write $n^4 \equiv 0(mod16)$

n$^4 = 16or$n$\equiv 1(16)$ n$^4 - 1 = 16$

n$^8 \equiv 0, 1(mod32)$

$n^8 \equiv 0(mod32)$

n$^8 = 32k$

$n^8 \equiv 1(mod32)$

n$^8 - 1 = 32k$

k are constant number , as according to n odd or even

$\tau(n) - n^3\sigma_1(n) \equiv 0(mod16)$;

$\tau(n) - n^3\sigma_5(n) \equiv 0(mod32)$.

Ramanujan discovered congruences like:

p(n5+4) $\equiv 0(mod5), p(7n + 5) = 0(mod7), p(11n + 6) \equiv 0(mod11)$

These results are foundational in understanding the arithmetic of partition functions and have inspired vast generalizations in modular forms and q-series

the Ramanujan divisibility $\tau(n)$ are uesd in Partition Theory Number Theory or Cryptography Computational Number Theory.

5 References

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